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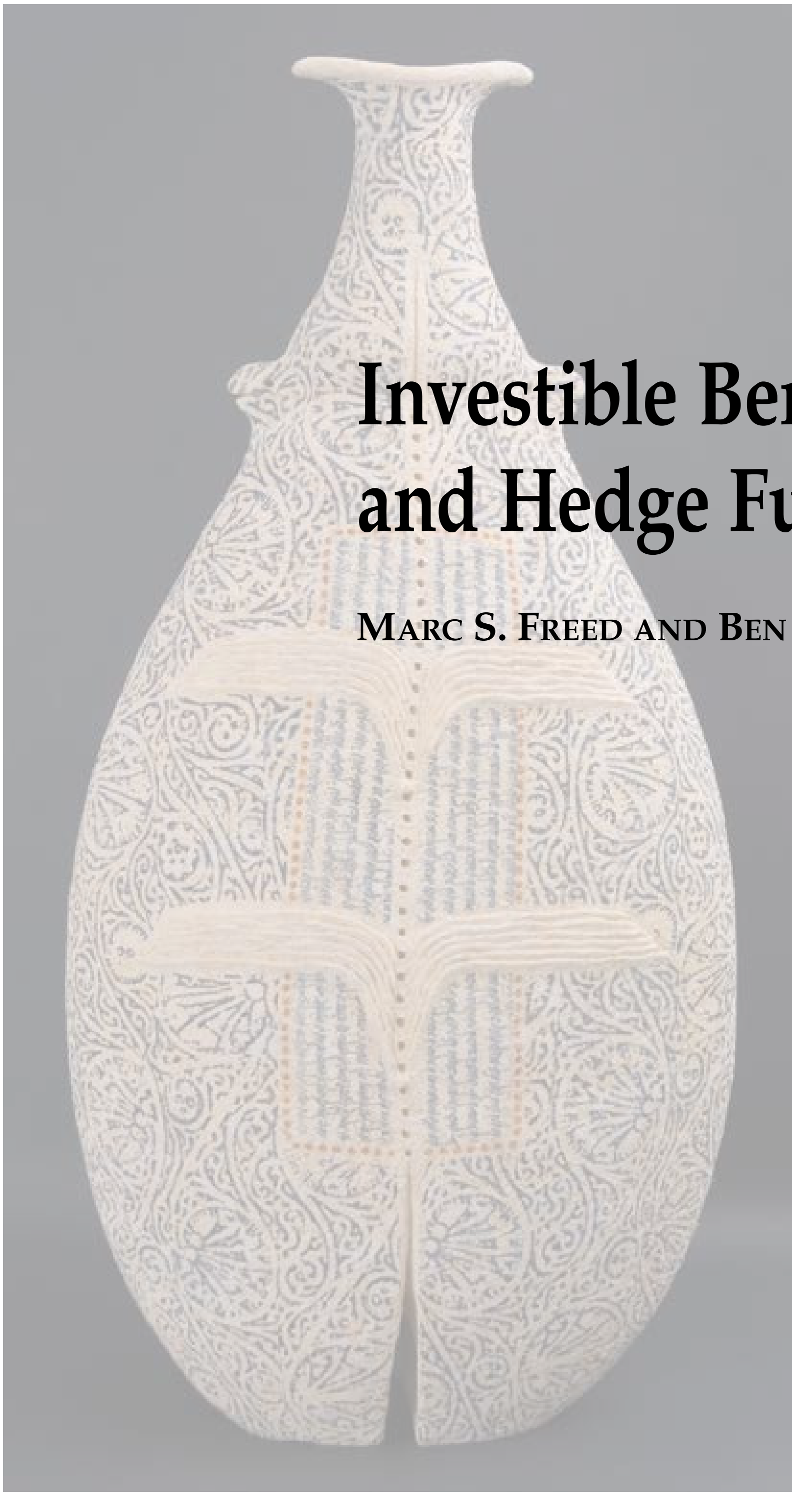
"Reality demonstrates that the more specific rules are, the easier, paradoxically, they make it to navigate around them. Moving ethics out of the realm of values and into the compliance and regulatory space has effectively failed us."

Jean L.P. Brunel – Editor's Letter



Investible Benchmarks and Hedge Fund Liquidity

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Marc S. Freed and Ben McMillan

Marc S. Freed
is portfolio manager at Van Eck Absolute Return Advisers Corporation in New York, NY.
mfreed@vaneck.com

Ben McMillan
is assistant portfolio manager at Van Eck Absolute Return Advisers Corporation in New York, NY.
bmcmillan@vaneck.com

Good investments outperform relevant benchmarks. Thanks to Treynor [1965], Sharpe [1966], and Jensen [1969] who pioneered that field of study decades ago, investors in conventional equity investments have well-established means of testing the hypothesis that truism implies.

ORIGINS OF PERFORMANCE BENCHMARKS

Largely because of their work, investors in equity mutual funds compare their results with investible performance indices of the markets in which their funds invest. Research by Fisher [1930], Markowitz [1953], Tobin [1958], and Sharpe [1964] established comparable paradigms for fixed-income investments. As a result, bond investors measure nominal returns against historic term and risk premia and real returns against inflation.

Alternative assets lack well-defined and widely accepted performance benchmarks. Consider hedge funds. The many indices that track hedge fund performance suffer a common set of problems that limits their utility as standards of performance. These include survivor and backfill biases in databases, a lack of consensus on classification, and the inability to invest in an index proxy. This article argues that the recent growth of investible factor-based hedge fund replication vehicles begins to solve this problem. It also claims that the liquidity of such vehicles creates a framework for quantifying the price of hedge fund illiquidity.

DECOMPOSING HEDGE FUND RETURNS

In the absence of a consensus on accurate benchmarks for hedge funds, most consultants and professional allocators have limited their correlation analysis to broad long-only benchmarks, such as equity indices. This move has allowed hedge fund managers to claim that returns not correlated with these broad indices constitute evidence of their skill. They define these idiosyncratic returns as alpha.

Recent developments may alter standards for analyzing hedge fund returns, at least for some hedge fund strategies. For years, some analysts have reported and some investors have recognized the magnitude of hedge fund returns attributable to measurable risk factor exposures. Brown and Goetzmann [2001], Fung and Hseih [1997, 2000, 2004], Al-Sharkas [2005], and others have documented this extensively. Often called exotic beta, hedge fund beta, or something similar, these statistical artifacts have served to date primarily as measures of correlation that true believers shun in a quixotic quest for the elusive alpha particle that allegedly shares the same subatomic return space.

At the most general level, then, hedge fund returns comprise some idiosyncratic returns, some known and measurable returns, and some other "stuff" that in a linear regression of hedge fund returns and risk factors appears as statistical noise.

For a single hedge fund, we may describe this more formally as

$$R^f = \alpha^f + B^f X_T + \varepsilon^f \tag{1}$$

where

$$B^f = \begin{bmatrix} \beta_1^f & \beta_2^f & \dots & \beta_n^f \end{bmatrix} \tag{2}$$

and

$$X_T = \begin{bmatrix} X_T^1 \; X_T^2 & \dots & X_T^n \end{bmatrix}^T \tag{3}$$

In other words, the returns of a hedge fund comprise its uncorrelated non-random returns, the correlation-weighted non-random returns of n known risk factors computed over time period T, and some random returns. The correlations B^f apply only to the specific fund f while the regressors X_T represent the returns of a single set of risk factors to which all funds may exhibit sensitivity. In any period, correlation coefficients or regressors may hold positive or negative values. In terms of expectations, however, the standard linear regression model requires that the expected value of the random returns equal zero.

The regression described above produces estimates of a fund's α^f and B^f from time $t = 0$ to time $t = T$. Conventionally, one would interpret B^f as a measure of the correlation of returns attributable to known variables; $\alpha^f > 0$ would indicate that the manager has demonstrated some skill, and $\alpha^f < 0$, a lack thereof. A skeptic may question the assertion that positive *alpha* equals skill, because it may simply correspond to risk factors excluded from the regression. Such a claim may overstate a manager's insight and undervalue the portfolio management skills that determined its risk factor exposure.

Using this information to establish a benchmark holds more promise for fund evaluation. It requires only a small but subtly different interpretation of the regression results. From this perspective, we consider the quantity $B^f X_T$, a return available from known risk factors and, as such, a benchmark for the period from 0 to T. To observe the over- or underperformance of the fund, we rearrange (1) to obtain

$$\alpha_T^f + \epsilon_T^f = R_T^f - B^f X_T \tag{4}$$

We measure the difference between the return of the fund and the benchmark only in terms of its random and non-random components. We do not define any of it as skill, or a lack thereof, because true skill encompasses the ability to manage a portfolio's risk factor allocation as well as its idiosyncratic security selection. Since $E[\epsilon_T^f] = 0$, a fund outperforms its benchmark only when $\alpha_T^f > 0$ or

$$R_T^f > B^f X_T \tag{5}$$

Using such a proxy as a benchmark for the correlated portion of a fund's expected return establishes it as a floor on the minimum return an investor should expect from a fund over a time period of length T. In practice, this approach still has only limited value, because most individual funds exhibit low correlations with risk factors. Moreover fund-specific benchmarks do not permit us to compare funds fairly with each other.

CONSTRUCTING STRATEGY BENCHMARKS WITH RISK FACTORS

In their efforts to avoid correlation with known betas, hedge fund analysts and investors have used regression primarily to estimate correlation while overlooking the possibility of using the identifiable components of fund returns as benchmarks. Until recently, they may have done so as well because one could not invest directly in many of the risk factors identified in regression analyses of hedge fund returns. Absent the ability to invest in risk factors correlated with hedge fund returns, analysts reasonably declined to cite them as benchmarks for hedge fund performance.

By themselves, even investible risk factor proxies function poorly as benchmarks. Comparing hedge fund returns to those of multiple risk factors creates a one-to-many relationship between a hedge fund and a set of risk factor proxies. Knowing a fund's correlations with a range of risk factors tells us something about its historic exposures but little, in aggregate, about its performance relative to its peers. Unfortunately, such analysis adds more confusion than insight to the process of performance assessment.

Bundling risk factor allocations according to their degree of correlation with large universes of hedge funds mitigates this problem to a significant extent. The recent proliferation of listed ETFs, ETNs, and futures contracts has expanded the universe of investible risk factor proxies and accordingly reduced the constraint on using them to construct legitimate benchmarks for hedge fund returns. Following Hasanhodzic and Lo [2007], we may construct such a benchmark from investible risk factor proxies by weighting them by their correlation with the fund's returns over a specified period.

RISK FACTOR CORRELATION AT THE STRATEGY LEVEL

We characterize funds with similar correlations to common sets of risk factors as hedge fund strategies. Applying the same regression methodology described above to such strategy universes produces robust correlations for most of the major categories into which we tend to classify hedge funds. For clarity, we rewrite the equations above with slightly different notation. The regression becomes

$$R^s = \alpha^s + B^s X_T + \in^s \tag{6}$$

where

$$B^s = \begin{bmatrix} \beta_1^s & \beta_2^s & \dots & \beta_n^s \end{bmatrix} \tag{7}$$

While identical to Equations (1) and (2) but for a change in notation, the results of this regression require a different interpretation from that given for Equation (1). The regression results

$$\widehat{B}^s = \begin{bmatrix} \widehat{\beta}_1^s & \widehat{\beta}_2^s & \dots & \widehat{\beta}_n^s \end{bmatrix} \tag{8}$$

quantify the average exposure of all funds in the universe to a common set of risk factors. As such, they identify the factors that dominate the aggregate behavior of the funds in the universe of similar funds. At the same time, they highlight the insignificance to the universe as a whole of factors that may explain a lot of the behavior of only a few of the funds in the universe.

We measure the goodness-of-fit of a regression, roughly the percentage of the behavior of strategy returns R^s explained by the risk factors $\widehat{B}^s X_T$, by its so-called R-squared coefficient. Exhibit 1 shows the R-squared values of regressions of returns of some major hedge fund strategies against baskets of investible risk factor proxies for two periods, January 1999 to July 2011 and January 2007 to July 2011. The former covers periods of crises[1] and calm, while the latter covers only the period of the ongoing financial crisis.

The data suggests clearly that portfolios of risk factor proxies explain significant percentages of the returns of some strategies and insignificant portions of others. The increase in correlations in the most credit-sensitive strategies since 2007 suggests further that hedge fund beta varies directly with volatility, not inversely, as investors would prefer in products promising to deliver absolute returns. The apparent insensitivity of directional macro, statistical arbitrage, and commodity trading advisors (CTAs) to the credit crisis lends support to their claims of low correlation with common risk factors and to the suggestion herein that they do not lend themselves to replication.

Funds in strategies with high correlations to investible risk factors tend to have two characteristics in common: a concentration on corporate securities and a low rate of monthly portfolio turnover. In general, we observe that long/short equity and corporate credit strategies exhibit high correlations with investible risk factors. In contrast, relative value and arbitrage

EXHIBIT 1
Returns Attributable to Investible Risk Factors

Hedge Fund Strategy[2]	R-squared[3] Jan 1999–Jul 2011	R-squared[3] Jan 2007–Jul 2011
Long/Short Equity North America	95%	95%
Event Equity	89%	93%
Long/Short Equity Emerging Markets	84%	88%
Distressed Investment	79%	84%
Credit Strategies	78%	78%
Convertible Bond Arbitrage	62%	73%
Market-Neutral Equity	58%	61%
Directional Macro	55%	54%
Statistical Arbitrage	53%	54%
CTAs	34%	36%
Volatility Arbitrage	13%	37%

strategies, especially those focused on interest rates and foreign exchanges and those with rapid turnover, such as directional macro and statistical arbitrage, exhibit much less correlation with such instruments.

For highly correlated strategies, then, we claim that investible portfolios of correlation-weighted risk factor proxies constitute ex post benchmarks for significant percentages of the relevant strategies' expected returns. Specifically, to the extent that a strategy's aggregate beta $\widehat{B}^s X_T$ accounts for a relatively stable percentage of its returns over time, a time series of returns of a portfolio designed to mimic that beta constitutes a lower bound on the performance one might reasonably have expected to earn from investments in that strategy over that period.

USING REPLICATION BENCHMARKS TO PRICE HEDGE FUND LIQUIDITY

Accounting for the explanatory variables of a regression of strategy level returns still leaves us without an explanation for the α^s, that is, the non-random portion of the strategy returns not correlated with known risk factors. Clearly, it does not describe skill since we cannot apply such a concept in the aggregate, a point that should raise questions about its use for that purpose at the fund level. Excluding skill as a possible meaning for α^s requires us to develop alternative explanations for it. By definition, we know that it may simply refer to unidentified risk factors not included among the X_T. Until we can identify them, we cannot incorporate them into the benchmark. An insight into the features of an investible benchmark suggests a more tangible and significant alternative explanation.

By their construction, factor-based replication funds lack the ability to generate alpha. Unlike hedge funds, they can explain 100% of their returns at all times in terms of the performances of the risk factor proxies in which they invest. They differ from the hedge funds whose strategies they attempt to mimic in other ways as well. Specifically, they can offer investors daily mark-to-market transparency and liquidity. A large investor in a replication fund that invests only in listed ETFs, ETNs, and futures contracts, for example, can close out such an investment in no more than a few hours. Smaller ones can convert their investments to cash in a matter of minutes.

In contrast, few hedge funds offer their investors even monthly liquidity. Most offer quarterly liquidity with significant notice periods. Some place even stricter limits on withdrawals. Many impose lockup periods of one or more years on new investments. Some permit early redemptions only with the payment of an exit fee. At the same time, managers retain the right to return capital to investors at any time.

Chacko [2005] found evidence that liquidity premia account for statistically significant portions of returns of corporate bonds. In his analysis, he introduced a concept of latent liquidity or accessibility that depended essentially on the willingness of bondholders to sell, what one might also call behavioral liquidity. In contrast, this article examines contractual liquidity between a hedge fund investor and a hedge fund manager. To reconcile these two phenomena, we recognize behavior as a contract with oneself. For example, an insurance company with an investment policy that limits reviews of holdings to quarterly periods has essentially imposed on itself a policy of quarterly liquidity.

The obvious differences in liquidity between factor-based replication funds and hedge funds suggest that a liquidity premium accounts for at least some portion of α^s, the excess returns observed at the strategy level. Surely a hedge fund investor subject to limited liquidity should earn more than an investor in a fund with more frequent liquidity that replicates only that hedge fund's known risk factors. Excess returns α^s observed at the strategy level, then, should include both idiosyncratic performance and compensation for the liquidity that hedge fund investors forgo relative to investors in replication funds.

Chacko, Das, and Fan [2011] model ex post behavioral illiquidity as a sum of two American options, one call and one put. To answer the ex ante question of how much of a liquidity premium an investor should expect to earn in exchange for entering into a contract that limits liquidity for the convenience of a hedge fund manager, we present an option-based model similar to those of Longstaff [1995] and Koziol and Sauerbier [2007] that contrasts the quasi-continuous liquidity of equity markets with the scheduled liquidity of hedge funds. Like Longstaff and Koziol and Sauerbier, our model posits a perfectly liquid proxy against which we compare a nearly identical asset with intermittent liquidity.

Continuous liquidity in a replication fund corresponds to the right to sell the fund at any time.

In option terms, we may describe this as the right to put the replication fund back to the market continuously. Mathematically, we may express this value as a sum of an infinite series of at-the-money put options of infinitesimal duration.[4] Let λ^s represent the value of continuous liquidity available to a replication fund investor.

$$\lambda^s = \int_0^\infty P^s(t) \tag{9}$$

Evaluating this integral lies beyond the scope of this article. Fortunately, we know from elementary calculus that it equals or exceeds the value of a sum of discrete at-the-money options of measurable duration, that is,

$$\int_0^\infty P^s(t) \geq \sum_{t=0}^{T} P^s(t) \tag{10}$$

We use the right-hand term of this equation to estimate the value of the liquidity of a replication fund offering daily liquidity.

In contrast to the replication fund, we may value the periodic liquidity of a hedge fund as a single European put option expiring on the fund's redemption date. We let λ^f represent the value of this option ex ante, that is, at the time of investment.

$$\lambda^f = P^f(T) \tag{11}$$

Because we cannot know the value of the fund at time T, we cannot know ex ante the true strike price of the single hedge fund redemption option. Assuming that funds produce positive returns more often than not, we may model such options as in-the-money instead of at-the-money puts struck at the expected value of the hedge fund at time T. To produce a conservative estimate of the value of the liquidity option that hedge fund investors grant to their managers, we assume a strike price for the European option based on an assumed positive return for the hedge fund.

To estimate the value of the liquidity premium that a hedge fund with liquidity available only at time T collects from its investors in comparison to that of a replication fund offering continuous liquidity, we simply take the ex ante difference between the two option values.

$$\Lambda^T = \lambda^s - \lambda^f = \sum_{t=0}^{T} P^s(t) - P^f(T) \tag{12}$$

This expression captures the value to an investor exchanging daily liquidity for a single fixed redemption date.

ESTIMATED VALUES OF HEDGE FUND ILLIQUIDITY

We use the standard discrete Black–Scholes option-pricing model to price European puts of different durations. To value the sum of one day at-the-money puts, we discount each one by the appropriate forward rate so as not to overstate the combined value of the puts. Of course, the term put $P^f(T)$ requires only a single, straightforward calculation. The model produces some startling results when one compares the value of daily liquidity to that of almost any other term. They appear reasonable only when one compares longer redemption periods with each other. Significantly, our results resemble those of Longstaff [1995], who commented in his article, "discounts for lack of marketability can be large even when the length of the marketability restriction is very short."

To compare different liquidity regimes, we compare different liquidity preferences with different liquidity profiles. For example, we compute the value to an investor seeking monthly liquidity of investments with quarterly and annual liquidity. This approach allows us to map the liquidity premia that investors should demand for investments that exceed their preferred liquidity.

We assume annualized volatility of 8% for both funds, a number similar to the long-term level observed for funds that replicate the performance of U.S.-focused long/short equity hedge funds, and an annual riskfree rate of 2%, a more realistic estimate than that imposed by current U.S. Federal Reserve policy. The assumption of equal volatility for the replication fund and the hedge fund makes sense because the risk factors captured in the replication fund account for such high percentages of the returns of the hedge fund. To estimate the strike prices of the term puts on the hypothetical hedge fund, we assume an expected annual return of 10% for the hedge fund. In addition, to allow for the possibility that the hedge fund produces non-random returns in excess of the benchmark (i.e., alpha), we assume an expected return of only 8% for the replication fund. This automatically reduces the liquidity premium in the hedge fund's favor.

Using parameters that favor the fund with limited liquidity over the fund with high liquidity does not produce liquidity premia that one might reasonably expect to recover in exchange for surrendering a large amount of one's preferred liquidity. Such premia appear attainable only when one compares preferred liquidity of no less than monthly frequency with actual liquidity of semi-annual or annual frequency. The semi-annual/annual comparison is negative only because of the assumption that the less-liquid fund will outperform the more-liquid one by 2%. An assumed difference of only 1% produces a liquidity premium of 0.12%, while an assumption of equal returns implies a liquidity premium of 0.83%

Because option prices vary directly with volatility, the liquidity premia between benchmarks and the assets they measure shrink if the benchmarks have lower volatility than the benchmarked assets. Reasonable differences in volatility alone cannot eliminate completely, however, the liquidity premia shown in Exhibit 2. For example, reducing the daily/quarterly premium from 13.15% to 0% while holding the hedge fund volatility steady at 8% requires a benchmark volatility of 1.02%. While instruments with such low volatility exist, they make poor benchmarks for hedge funds in comparison with the factor-based replication funds discussed herein. Thus, differences in volatility alone cannot account for the differences in liquidity between replication benchmarks and individual hedge funds.

To conclude, the incremental returns required to make an investor indifferent between two investments that differ significantly only in their liquidity terms seem attainable only for investors already content with limited liquidity. A fund manager might earn an additional 0.82% per quarter to compensate an investor who prefers monthly liquidity for accepting quarterly liquidity, but no one should expect to earn an additional 55% by exchanging daily for annual liquidity.

The extreme differences between the value of daily liquidity and all other redemption terms raise serious questions about the practice of funds offering less liquidity than those of the assets in which they invest. Specifically, with respect to funds that replicate hedge fund returns, redemption restrictions managers place on their funds undermine their claims that skill rather than illiquidity accounts for their returns in excess of a replication benchmark.

CONCLUSIONS

The contrasting results of risk factor regressions on individual funds and on collections of funds classified as strategies force hedge fund investors to assess their commitment to the paradigm of strategy classification of hedge funds. An investor who views hedge funds as unique investment vehicles for which strategy classifications serve merely an accounting function may discount the value of bundles of risk factor proxies. In contrast, an investor who acknowledges that strategy classifications properly reflect similarities in the behavior of many hedge funds may embrace the notion of factor-based replication vehicles as benchmarks for their allocations to the corresponding hedge fund strategies.

Recognizing factor-based replication funds as legitimate benchmarks permits us to view the uncorrelated non-random portion of strategy level returns as a proxy for differences in liquidity between replication funds and the funds whose correlated non-random returns they emulate. Using an option model, we observe that daily liquidity has a value far in excess of any excess returns one might expect to earn from most hedge funds with limited liquidity.

These results suggest that hedge fund managers in strategies that investors can emulate to a significant extent with more liquid alternatives may need to relax their redemption terms as replication funds grow in popularity. More significantly, it implies that most hedge fund returns come not from manager skill but from the value of options that

EXHIBIT 2

Yield Compensation for Less-than-Preferred Liquidity

% per Deferral Period		Preferred Liquidity: Daily	Weekly	Monthly	Quarterly	Semi-Annual	Annual
Actual Liquidity	Daily	0					
	Weekly	0.71%	0				
	Monthly	3.98%	0.88%	0			
	Quarterly	13.15%	3.76%	0.82%	0		
	Semi-Annual	27.07%	8.30%	2.41%	0.14%	0	
	Annual	54.65%	17.43%	5.65%	1.11%	−0.56%	0

investors, when they invest in a hedge fund, sell implicitly to fund managers at prices far below their market values.

APPENDIX

DERIVING LIQUIDITY PREMIA FROM OPTION PRICES

Since the arrival of the Black–Scholes option-pricing model [1972], practitioners of academic and applied finance have focused their research on volatility and the effects of irregular cash flows. Invariably, they have viewed *time*, one of the five variables required to price options, as given, or in comparing option prices for different assets, as equal for each asset.

In this article, we argue that an investment manager may claim to have produced excess returns over a benchmark only after compensating investors for any differences in liquidity between the managed asset and the benchmark. The main body of the article contains the argument for treating an investible replication fund as a benchmark but presents only an abridged justification for using the option-pricing model to evaluate the differences in liquidity between a benchmark fund and a hedge fund for which it serves as a reference. This Appendix contains a more complete explanation of this approach.

To begin, we invert the normal representation of options as derivatives of some financial asset. We argue, instead, that financial assets themselves lack inherent existence. They derive their value from a continuous series of at-the-money options on the real assets whose values they embody. From this perspective, we observe that an exchange-traded equity, for example, comprises an infinite series of at-the-money calls and puts on some real assets. *Stock* is merely a name that we impute to this collection of options. We define this phenomenon as follows:

Definition 1: *A financial asset comprises infinite series of at-the-money calls and puts of infinitesimal duration. All such series have lower bounds of t = 0. Series for assets with final maturity dates have upper bounds of t = T; series for assets without a final maturity date have no upper bound.*

Formally, we may describe the unbounded scenario with the equation

$$s = \int_0^\infty C(t) + \int_0^\infty P(t) \tag{A1}$$

Most readers will see instantly how the right of an owner of a share of stock to sell it at any time corresponds to a continuous series of at-the-money put options; but where, one may wonder, are the call options? To see this, we introduce two rules: the long rule and the short rule.

Long Rule: *A long position in a financial asset remains a long position until and unless the owner exercises one of the infinite series of put options available to terminate the long exposure. Such put options may be available continuously or intermittently. The asset owner implicitly exercises each of the infinite series of calls at every moment that it does not choose to exercise a put to negate the long exposure.*

Short Rule: *A short position in a financial asset remains a short position until and unless the owner exercises one of the infinite series of call options available to terminate the short exposure. Such call options may be available continuously or intermittently. The holder of the short position implicitly exercises each of the infinite series of puts at every moment that it does not choose to exercise a call to negate the short exposure.*

We may incorporate these rules into Definition 1 to define long and short positions in terms of the characteristics of the options they comprise.

Definition 1a: *A long position in a financial asset comprises an infinite series of at-the-money call options of infinitesimal duration and put options of infinitesimal or intermittent duration with the call options deemed exercised automatically, unless the holder exercises one of the put options to liquidate the position.*

Definition 1b: *A short position in a financial asset comprises an infinite series of at-the-money put options of infinitesimal duration and call options of infinitesimal or intermittent duration with the put options deemed exercised automatically, unless the holder exercises one of the call options to cover the position.*

From this perspective, both owners and short sellers of assets buy options continuously. Long holders pay their option premia implicitly with a combination of the forgone interest on the cash paid for the asset and some portion of its future returns. Ceteris paribus, a long position has a gain when

$$S_{t+\eta} > W_t \rightarrow \pi > 0 \tag{A2}$$

where S equals the market price of the real assets underlying the options at time $t + \eta$, W_t the strike price of the at-the-money options at time t, π the profit on the position, and η an infinitesimal amount of time past t. Conversely, long positions lose value when

$$S_{t+\eta} > W_t \rightarrow \pi < 0 \tag{A3}$$

Clearly, the reverse conditions apply to short positions. Thus, in this paradigm, returns on financial assets equal the aggregate differences between the value of all the options they comprise from the time one acquires them (or establishes a short position in them) and the cost of those options that expire while one owns them (or maintains a short position in them).

To apply this analysis to the benchmarking example at issue in this article, we let S_b represent the price (normalized net asset value) of the benchmark replication fund and S_f the price (normalized net asset value) of a hedge fund claiming to offer returns in excess of the benchmark. Because the volatility of the investible benchmark and the funds it references should be similar, we assume that only redemption terms differentiate S_b from S_f, with the former offering continuous liquidity and the latter liquidity only on a specific date T. Then, following Equation A1, we may describe the price of the benchmark fund with the following:

$$S_b = \int_0^\infty C(t) + \int_0^\infty P(t) = \int_0^\infty C(t) + \int_0^T P(t) + \int_{T+\eta}^\infty P(t) \qquad \text{(A4)}$$

We describe the hedge fund as:

$$S_f = \int_0^\infty C(t) + P(T) + \int_{T+\eta}^\infty P(t) + \Lambda_L^T \qquad \text{(A5)}$$

where Λ_L^T represents the value of the liquidity differential between a long position in a benchmark with continuous liquidity and the hedge fund that offers liquidity initially only at time T. For the hedge fund to perform at least as well as its benchmark, we must have

$$S_f \geq S_b \rightarrow S_f - S_b \geq 0 \qquad \text{(A6)}$$

Substituting Equations A4 and A5 into Equation A6 and simplifying the resulting equation produces

$$P(T) + \Lambda_L^T - \int_0^T P(t) \geq 0 \qquad \text{(A7)}$$

$$\Lambda_L^T \geq \int_0^T P(t) - P(T) \qquad \text{(A8)}$$

Thus, the liquidity premium for a long asset with restricted liquidity, like a hedge fund, over a nearly identical asset redeemable continuously must at least equal the value of the continuous infinitesimal at-the-money put options expiring from inception of the position until time T less the value of a single European put option expiring at time T.

A similar analysis comparing a short asset with continuous liquidity with another short asset offering only a single opportunity to cover the short produces a value derived from option prices of

$$\Lambda_S^T = \int_0^T C(t) - C(T) \qquad \text{(A9)}$$

We examined the benchmark asset under two scenarios: as a standalone entity (i.e., a replication fund) with continuous liquidity and as a benchmark embedded in a hedge fund. The liquidity valuation hypothesis assumes only that the hedge fund's volatility equals that of its benchmark. To analyze the hypothesis, we compare an at-the-money European put option on the benchmark fund with a single in-the-money European put option on the hedge fund. We used the Black–Scholes option model to price the liquidity of each scenario. To make the option prices equivalent to percentage values, we set the underlying price of each benchmark to 100. We used a constant volatility of 8%, slightly more than that of the 10-year historical level of the benchmark derived from a backtest of hedge fund data. We set the riskfree rate at 2%, a more historically reasonable level than that of the post-2008 central bank regime.

Because these two calculations produce different results and because differences in permissible times for execution correspond to different liquidity profiles, we interpret price differences between options resulting exclusively from differences in the time variable of the option-pricing model as reasonable estimates of the relative value of the differences in liquidity schedules.

In analyzing the benchmarking example in this article, we have assumed that the volatility of the benchmark equals that of the assets with which one would compare it. While this makes sense in the hedge fund example, it need not be true in general. This approach to pricing liquidity should apply to assets that differ in both the frequency with which they trade and in volatility. As we mentioned in the body of the article, the values of liquidity differentials shrink dramatically when one compares liquid assets with low volatility to illiquid ones with high volatility. In the context of hedge funds and the emerging market for investible benchmarks, however, such differences in volatility exceed by far the levels required to justify the limited liquidity most hedge funds offer their investors.

ENDNOTES

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[1]This period encompasses the climax and collapse of the Internet bubble from 1999–2001, the corporate-fraud credit crisis of 2002, the benign period from 2003–2006, and the current financial crisis that began in 2007.

[2]The authors have classified funds in the hedgefund.net™ database into the listed strategies based on the commonality of their risk factor correlations.

[3]Results are based on regressions of monthly performance for all funds in the universes for any consecutive 12-month period for the 2 periods shown below against 23 tradable risk factor proxies drawn from the universes of ETFs, government bonds, commodity futures contracts, and foreign exchanges.

[4]The Appendix contains a detailed explanation of the theory that allows us to estimate liquidity premia with option prices.

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Addendum:The Equivalence of Equity Prices and At-the-money Straddles

In a paper entitled "Investible Benchmarks and Hedge Fund Liquidity" that my colleague Ben McMillan and I published recently, we present an option model to quantify the value of iliquidity that hedge fund investors forfeit when they invest in hedge funds with restricted redemption schedules. The analysis in the paper depends on a statement that conventional financial assets such as stocks and bonds are themselves derivatives of options, not the reverse as usually presented in the standard teachings on options. This addendum to the paper attempts to provide a more formal explanation of this idea in a theorem that asserts an equivalence between a specific set of options and a non-dividend paying equity. This theorem establishes a multi-period no-arbitrage condition between options with the specified parameters and equities.

Theorem: *The limit of an infinite series of at-the-money straddles of infinitesimal duration on a non-dividend paying stock at time θ = 0 converges to the price of the stock at time θ = 0.*

$$\lim_{t \to 0} \int_{\theta=0}^{\infty} C\left(S_\theta, X_\theta, \sigma_\theta, r_\theta, t\right) d\theta + \lim_{t \to 0} \int_{\theta=0}^{\infty} P\left(S_\theta, X_\theta, \sigma_\theta, r_\theta, t\right) d\theta = S_\theta$$

Proof: To show this to be true we must decompose the statement above into three parts: the normal distribution parameters used to compute the option prices, the option prices themselves, and the infinite series. The normal distribution parameters embedded in the option price formulae are functions of all of the variables used to compute the option prices. Because this theorem applies only to European at-the-money options, we treat as constants at each moment θ the price of the underlying stock S_θ, the at-the money strike price $X_\theta = S_\theta$, the volatility of the stock price σ_θ, and the short-term risk free interest rate r_θ.
Because we are talking about options with infinitesimal durations we write the time variable as $t + \varepsilon$ where ε represents an infinitesimal amount of time between an option's inception and its expiration.
Each option price includes the cumulative probability distribution values associated with the stock price $N(d1)$ and the discounted values of the strike price $N(d2)$ where $d1$ equals

$$\frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)(t+\varepsilon)}{\sigma\sqrt{t+\varepsilon}} \tag{1}$$

Take the limit of $N(d1)$ as $t \xrightarrow{\text{limit}} 0$

$$\frac{1}{2}\,\frac{2\ln\left(\frac{S}{X}\right) + 2r\varepsilon + \sigma^2\varepsilon}{\sqrt{\varepsilon}\,\sigma} \tag{2}$$

Evaluate the limit at $S = X \xrightarrow{\text{evaluate at point}}$

$$\frac{1}{2}\,\frac{2\,r\,\varepsilon+\sigma^{2}\,\varepsilon}{\sqrt{\varepsilon}\;\sigma} \tag{3}$$

and simplify the resulting expression. $\overset{\text{simplify radical}}{=}$

$$\frac{1}{2}\,\frac{\sqrt{\varepsilon}\;(2\,r+\sigma^{2})}{\sigma} \tag{4}$$

Similarly for $N\,(d2)$ we have

$$\frac{\ln\left(\frac{S}{X}\right)+\left(r-\frac{1}{2}\,\sigma^{2}\right)\,(t+\varepsilon)}{\sigma\,\sqrt{t+\varepsilon}} \tag{5}$$

Take the limit as $t \xrightarrow{\text{limit}} 0$

$$\frac{1}{2}\,\frac{2\,\ln\left(\frac{S}{X}\right)+2\,r\,\varepsilon-\sigma^{2}\,\varepsilon}{\sqrt{\varepsilon}\;\sigma} \tag{6}$$

Evaluate this limit at $S=X \xrightarrow{\text{evaluate at point}}$

$$\frac{1}{2}\,\frac{2\,r\,\varepsilon-\sigma^{2}\,\varepsilon}{\sqrt{\varepsilon}\;\sigma} \tag{7}$$

and simplify the resulting expression. $\overset{\text{simplify radical}}{=}$

$$\frac{1}{2}\,\frac{\sqrt{\varepsilon}\;(2\,r-\sigma^{2})}{\sigma} \tag{8}$$

Take the difference between the limits of $N\,(d1)$ and $N\,(d2)$.

$$\frac{1}{2}\,\frac{\sqrt{\varepsilon}\;(2\,r+\sigma^{2})}{\sigma}-\frac{1}{2}\,\frac{\sqrt{\varepsilon}\;(2\,r-\sigma^{2})}{\sigma} \tag{9}$$

to see the relationship between them. $\xrightarrow{\text{assuming real}}$

$$\sqrt{\varepsilon}\;\sigma$$

Since $\varepsilon > 0$ and $\sigma > 0$ the quantity $\sqrt{\varepsilon}\cdot\sigma > 0$. From this we can see that $N\,(d1) \geq N\,(d2)$ and $N\,(-d2) \geq N\,(-d1)$. In words, as long as the options have some albeit infinitesimal lifespan, the difference between the two distributions will be positive. This will be important in the next step of this proof in which we take the lmits of the option prices themselves.

Using the Black-Scholes option pricing formula, we compute the following limit for at-the-money calls and puts at time θ as the terms of the options t approach zero.

The price of a European call $C = S \cdot N(d1) - X \cdot e^{-r \cdot t} \cdot N(d2)$

$$S\,N(d1) - X\,e^{-r\,t}\,N(d2) \tag{10}$$

Take the limit of the call price as $t \xrightarrow{\text{limit}} 0$

$$S\,N(d1) - X\,N(d2)$$

Since we are concerned only with at-the-money options we substitute $S = X$ and factor that result to obtain

$$\xrightarrow{\text{evaluate at point}}$$

$$S\,N(d1) - S\,N(d2) \overset{\text{factor}}{=} S\,(N(d1) - N(d2))$$

Similarly for an at-the-money put $P = X \cdot e^{-r \cdot t} \cdot N(-d2) - S \cdot N(-d1)$

$$X\,e^{-r\,t}\,N(-d2) - S\,N(-d1) \tag{11}$$

Take the limit of the put price as $t \xrightarrow{\text{limit}} 0$

$$X\,N(-d2) - S\,N(-d1)$$

As with the call price, since we are concerned only with at-the-money options we substitute $S = X$ and factor that result to obtain

$$\xrightarrow{\text{evaluate at point}}$$

$$S\,N(-d2) - S\,N(-d1) \overset{\text{factor}}{=} S\,(N(-d2) - N(-d1))$$

Now we recognize that

$$N(d1) - N(d2) = N(-d2) - N(-d1) \tag{12}$$

Having established that these differences always have a postive value for any $\varepsilon > 0$, we may let

$$\eta = N\left(d1\right) - N(d2) = N(-d2) - N(d1)$$

From this we see that the price of an at-the-money straddle of infinitesimal duration at any time θ equals

$$C_\theta + P_\theta = S \cdot \eta + S \cdot \eta = 2 \cdot S \cdot \eta.$$

Finally we may describe an infinite series of such straddles as $\sum_{n=1}^{\infty} \frac{2 \cdot S \cdot \eta}{(1+r)^{\frac{n \cdot \varepsilon}{365}}}$, a series that sums to the following quantity.

$$\frac{2\,S\,\eta}{-1+(1+r)^{\frac{1}{365}\varepsilon}} \tag{13}$$

In order for this expression to equal just *S*, we must satisfy the following statement.

$$\frac{2\,\eta}{(1+r)^{\frac{1}{365}\varepsilon}-1}=1 \tag{14}$$

$\xrightarrow{\text{manipulate equation}}$

$$\frac{2\,\eta}{(1+r)^{\frac{1}{365}\varepsilon}-1}=1$$

$$2\,\eta=(1+r)^{\frac{1}{365}\varepsilon}-1 \tag{15}$$

We have already established that both η and ε represent infniitesimally small positive values. Thus we may take the limits of each side of the equation as both η and ε approach zero.

Isolating the left side of the equation $\xrightarrow{\text{left hand side}}$

$$2\,\eta \tag{16}$$

we take its limit as $\eta \xrightarrow{\text{limit}} 0$

$$0 \tag{17}$$

Now we do the same to the right side of the equation.

$$2\,\eta=(1+r)^{\frac{1}{365}\varepsilon}-1 \tag{18}$$

$\xrightarrow{\text{right hand side}}$

$$(1+r)^{\frac{1}{365}\varepsilon}-1 \tag{19}$$

Its limit as $\varepsilon \xrightarrow{\text{limit}} 0$ is the same.

$$0 \tag{20}$$

Thus the theorem holds at the limit as the time until expiration of the options approaches zero.

Implications of the Theorem

In our paper we use this theorem as a philosophical justification for assigning a value to the redemption restrictions hedge funds impose on their investors. It may have implications beyond that for the study of investments and the business of investment management. For example, if we view markets as bundles of options rather than as bundles of stocks, bonds or commodities, then we can explain a portion of their daily gyrations as changes in the level of investor uncertainty about the known information pertaining to their value, and not as changes in known information itself. To the extent that rising uncertainty increases volatility, it increases the values of options that constitute the prices of financial assets. Because we discount the value of such options to obtain their present value, increasing the values of all such options while holding discount rates constant means that the options closest to expiration gain value at the expense of those more distant. Thus option decay, which occurs at a constant rate for options of infinitesimal duration but at a volume that varies with option values, causes prices to vary inversely with volatility, a phenomenon that we observe frequently in markets for financial assets.